MODAL INTERVALS : REASON AND GROUND SEMANTICS

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1. SEMANTIC RELATION BETWEEN INTERVALS AND REALS.

The structure STR(RE) of real numbers (would "ideal numbers" be a better name, to emphasize their being out of reach for digital computing?) , is the seat of geometrical intuition , which is based on the system of relations and operations that allow the construction of predicates P : RE --> SET(FALSE,TRUE) .

Otherwise, there is no computing system able to use the full STR(RE) and no finite digital set DI << RE (<< is "included in") closed for the whole system of exact arithmetical operations. This fact makes the structure STR(I(DI)) of digital intervals with outer rounding, into the only support for digital computing supplying the maximal approximating-information accessible to the system STR(DI) ; and STR(I(RE)) into its analytical frame.

The computing/analytic system STR(I(RE),I(DI)) displays, however , some critical problems ; let us look at three paradygmatic examples.

First : supposing F' <* I(RE) (<* is "belongs to") to be the exact result of an interval computation , its outer digital approximation FO >> F' (>> is "includes") keeps the validity of any predicate of the form E(f,F')P(f) when FO substitutes F' (E(f,F') is "exists f <* F' such that") ; but, in order to keep the validity of predicates of the form U(f,F')P(f) (U(f,F') is "for every f <* F' "), an inner digital approximation FI << F' ought to be used. The problem is that, though any F' <* I(RE) bounded by the system I(DI) admits an outer rounding FO, this property does not hold for the inner rounding FI (e.g. , the inner rounding of any x <* RE , x -<* DI , does not exist in DI (-<* is "does not belong to")).

Second : the lack of inner rounding in STR(I(RE),I(DI)) is also a drawback for the computation of the approximated interval solution of

I(RE)-systems like (A' + X' = B' , X' + Z' = C') where an inner rounded X' were needed to compute an outer rounded Z'.

Third : when the solution of the equation A' + X' = B' in I(RE) exists, the relation A' + X1' = B' is equivalent to the proposition " B' is the inclusion-least interval for which U(a,A') U(x,X1') a + x <* B' holds " ; but even when the interval solution of the equation A' + X' = B' fails to exist in the I(RE) context , an interval X" <* I(RE) does exist validating the proposition " B' is the inclusion-least interval for which U(a,A') E(x,X") a + x <* B' holds" .

These three "non-sequitur" situations stated in terms of the system STR(I(DI),I(RE)) , are evidence enough to undermine the validity of this system as universal frame for the numerical-computing theory.

To find out what is missing , let us analize the relation standing among intervals , real numbers , interval predicates , and predicates about real numbers.

Two-variable predicates like (P(x), x <* X') , ((pred,pred,...) is "pred AND pred AND ... ") , maybe would convey some P(.)-semantics from x <* RE to X' <* I(RE) , but the resulting semantics would be ambiguous for an interval argument X' because of the different truth values that P(.) could take for different points x <* X' ; moreover, this predicates would have only a designational value and would be out of question in a computational context , because of the unability to reach a general x <* X' by means of DI.

But classical interval-predicates can be obtained from real-predicates P(x) , without any reference to particular x <* RE , by means of the transformations P(x) --> E(x,X')P(x) and P(x) --> U(x,X')P(x) which transport the meanings defined by the predicates P(.) , from the domain RE to the domain I(RE).

Actually , the semantic transformation SEM : P(x) -> Q'(x,X')P(x) (Q' <* SET(E,U)) , brings predicates P(x) of the single real arguments x <* X' into predicates P*((X',Q')) := Q'(x,X')P(x) about the arguments X = (X',Q') <* I*(RE) , I*(RE) := CART(I(RE),SET(E,U)) (CART is "Cartesian product") , which we will name "modal intervals" (:= is "defined by") .

Indeed , if for X = (X',QX) we define SET(X) := X' and MOD(X) := QX (we will name SET(X) the set-component of X and MOD(X) its modality), the meaning of the predicate P*(X) = Q(x,X)P(x) is fully determined by the definition : Q(x,X) :=

(IF MOD(X) = E THEN E(x,X') ,
IF MOD(X) = U THEN U(x,X')) .

2. INTERVAL-SETS OF PREDICATES AND MODAL INCLUSION.

Let be PRED((X',QX)) := SET(P(.)/(Q(x,X)P(x)) the set of real predicates validated by the modal interval X = (X',QX) , and let us examine which are the conditions standing between the modal intervals A and B that correspond to the set inclusion PRED(A) << PRED(B).

LEMMA 2.1 PRED((A',E)) << PRED((B',E)) <==> A' << B'

Since A' << B' implies that (x1 <* A' , P(x1)) ==> (x1 <* B' , P(x1)) obviously ; and if A' -<< B' , E(a,A') a -<* B' and (x=a) <* PRED((A',E)) , but (x=a) -<* PRED((B',E)) and , therefore , PRED((A',E)) -<< PRED((B',E)).

LEMMA 2.2 PRED((A',U)) << PRED((B',U)) <==> A' >> B'

Since A'>>B' implies that U(x,A')P(x) ==> U(x,B')P(x) and if A' ->> B' , then E(b,B') b -<* A' and (x <* A') <* PRED((A',U)) but (x <* A') -<* PRED((B',U)) and therefore PRED((A',U)) -<< PRED((B',U)).

LEMMA 2.3 PRED((A',U)) << PRED((B',E)) <==> A' =* B'
(=* is "intersects")

Since A' =* B' implies that U(x,A')P(x) ==> E(x,B')P(x) ; and if A' -=* B' then (x <* A') <* PRED((A',U)) but (x <* A') -<* PRED((B',E)) and therefore PRED((A',U)) -<< PRED((B',E)).

LEMMA 2.4 PRED((A',E)) << PRED((B',U)) <==> A' = B' = INT(a)
(INT(a) is "the point-interval with a = inf = sup ")

Since , if a1 <* A' then (x=a1) <* PRED((A',E)) and the only possibility for the validity of U(x,B')(x=a1) is that B' = INT(a1) ; but in this case if a2 -= a1 , a2 <* A' , would exist , the predicate (x=a2) would be validated by (A',E) but not by (B',U) . The reverse implication is obvious .

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DEFINITION 2.1  For A = (A',QA) , B = (B',QB) modal intervals ,
    A << B  :=  IF  QA = QB = E  THEN  A' << B'
                IF  QA = QB = U  THEN  A' >> B'
                IF  ( QA = U , QB = E )  THEN  A' =* B'
                IF  ( QA = E , QB = U )  THEN  A' = B' = INT(a)
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DEFINITION 2.2  For A = (A',QA) <* I*(RE) ,
    INF(A) :=  IF  QA = E  THEN  INF(A')
               IF  QA = U  THEN  SUP(A')
    SUP(A) :=  IF  QA = E  THEN  SUP(A')
               IF  QA = U  THEN  INF(A')
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THEOREM 2.1 For A , B modal intervals ,
(INF(A) = INF(B) , SUP(A) = SUP(B)) <==> A = B

DEFINITION 2.3 For a , b <* RE ,
INT(a,b) := ELEM(A / A <* I*(RE) , INF(A) = a , SUP(A) = b)
(where ELEM(A / C) is the element named A fulfilling the condition C)

DEFINITION 2.4
Ie(RE) := SET((A',Q') / A' <* I(RE) , Q' = E)
Iu(RE) := SET((A',Q') / A' <* I(RE) , Q' = U)
Ip(RE) := SET((A',Q') / A' <* I(RE) , INF(A') = SUP(A'))

THEOREM 2.2 I*(RE) <--> SET((a,b) / a , b <* RE)
Ie(RE) = SET(A / A <* I*(RE) , INF(A) <= SUP(A))
Iu(RE) = SET(A / A <* I*(RE) , INF(A) >= SUP(A))
Ip(RE) = SET(A / A <* I*(RE) , INF(A) = SUP(A))

THEOREM 2.3 A <* I*(RE) ==> PRED(A) -= VOID
(VOID is "the void set")

Since , when A = (A',E) then (x = INF(A)) <* PRED(A) ,
and when A = (A',U) then (x <* A') <* PRED(A) .

And the above lemmata and definitions yield easily the following theorems for A , B , ... <* I*(RE) .

THEOREM 2.4 A << B <==> (INF(A) >= INF(B) , SUP(A) <= SUP(B))

THEOREM 2.5 A << B <==> PRED(A) << PRED(B)

THEOREM 2.6 A = B <==> (A << B , A >> B)
<==> PRED(A) = PRED(B)

Theorems 2.1 to 2.6 , by displaying the association (a1,a2) <--> PRED(INT(a1,a2)) , provide the lattice completion of the inclusion structure of ordinary intervals with a definitive semantical meaning , and suggest to interpret the elements of I*(RE) as acceptors/rejectors or interval-tests for the predicates about the reals , and to read "A << B" as "A is more strict than B" or "B is more tolerant than A" .

Maybe this semantics is clarified by the observation that for A <* I*(RE) , if A is a proper or exitencial modal interval (that is A <* Ie(RE)) then P(x) <* PRED(A) is equivalent to SET(x / P(x)) =* SET(A) , and if B is an improper or universal modal interval (that is B <* Iu(RE)) then P(x) <* PRED(B) is now equivalent to SET(B) << SET(x / P(x)) .

DEFINITION 2.5 For A <* I*(RE) ,
PROP(A) := INT(MIN(INF(A),SUP(A)) , MAX(INF(A),SUP(A)))
IMPR(A) := INT(MAX(INF(A),SUP(A)) , MIN(INF(A),SUP(A)))

The denomination PROP(A) comes from naming "proper intervals" the elements of Ie(RE) , or existencial intervals , because of their identification to the corresponding elements of I(RE) that arises from the equivalence in Ie(RE) of A << B and SET(A) << SET(B) . This identification keeps its force along the whole theory about I*(RE) , since the relation << in I*(RE) generates all the structure STR(I*(RE) , I*(DI)) . Moreover , the "proper intervals" are the interval-acceptors of the "exact" real solutions that the ordinary-interval approximations are meant to bound .

3. DUAL SEMANTICS OF MODAL INTERVALS.

We mean by dual semantics of modal intervals , their association to the real predicates of PRED(RE) they reject .

DEFINITION 3.1 COPRED(X) := SET(P(.) / -Q(x,X)P(x))

DEFINITION 3.2 DUAL(A) := INT(SUP(A),INF(A))

Essential theorems in this context are :

THEOREM 3.1 A <* I*(RE) ==> COPRED(A) -= VOID

THEOREM 3.2 COPRED(A) = PRED(RE) - PRED(A)

THEOREM 3.3 A << B <==> DUAL(A) >> DUAL(B)

THEOREM 3.4 P(.) <* COPRED(A) <==> -P(.) <* PRED(DUAL(A))

THEOREM 3.5 A << B <==> COPRED(A) >> COPRED(B)

THEOREM 3.6 IMPR(A) << PROP(A)

THEOREM 3.7 (A <* Ie(RE) , A -<* Ip(RE)) <==>
E(P(.) , PRED(RE)) (P(.) <* PRED(A) , -P(.) <* PRED(A))

THEOREM 3.8 For P(.) <* PRED(RE) and A <* I*(RE) ,
one of the two following alternatives holds
(1) (P(.) <* PRED(PROP(A)) , -P(.) <* PRED(PROP(A))) AND
(P(.) <* COPRED(IMPR(A)) , -P(.) <* COPRED(IMPR(A)))
(2) P(.) <* PRED(IMPR(A)) << PRED(PROP(A)) AND
-P(.) <* COPRED(PROP(A)) << COPRED(IMPR(A))

Perhaps it may be of some use to observe that for A <* I*(RE) , (A <* Ie(RE) , P(.) <* COPRED(A)) is equivalent to SET(A) -=* SET(x / P(x)) , and (A <* Iu(RE) , P(.) <* COPRED(A)) is equivalent to SET(A) -<< SET(x / P(x)) .

4. LATTICE SEMANTICS OF MODAL INTERVALS.

The structure STR(I*(RE) , <<) is isomorphic to the structure STR(CART(RE,RE)) , CART(>=,<=)) and , therefore , a distributive lattice like STR(RE , >=) and STR(RE , <=) . That is , given A , B <* I*(RE) , their <<-supremum or "join" JOIN(A , B) , and their <<-infimum or "meet" MEET(A , B) , do exist , and these operations are mutually distributive with the following operation laws :

MEET(A(i) / i <* I) :=
ELEM(A / U(i,I) (X << A(i)) <==> X << A) =
INT(MAX(INF(A(i)) / i <* I) , MIN(SUP(A(i)) / i <* I)
JOIN(A(i) / i <* I) :=
ELEM(A / U(i,I) (X >> A(i)) <==> X >> A) =
INT(MIN(INF(A(i)) / i <* I) , MAX(SUP(A(i)) / i <* I)

Now , for a full identification of the modal intervals A <* I*(RE) with the predicates-set PRED(A), it would be fine that PRED(JOIN(A,B)) would equal UNI(PRED(A),PRED(B)) , and that PRED(MEET(A,B)) wold stand in the same relation towards SEC(PRED(A),PRED(B)) ; where SEC and UNI stand for the set operations "intersection" and "union" .

This is far from certain , yet not so damaging to prevent a good semantical structure to hold on for the lattice of modal intervals .

To test this property we take , for example , the predicates-set PRED(MEET(INT(1,2),INT(3,4))) = PRED(INT(3,2)) . The predicate x <* SET(1.5,3.5) belongs to PRED(INT(1,2)) and to PRED(INT(3,4)) and therefore to the intersection of these two sets of predicates , but absolutely not to PRED(INT(3,2)) .

Also , x = 2.5 belongs to PRED(INT(1,4)) which is equal to PRED(JOIN(INT(1,2) , INT(3,4))) , but neither to PRED(INT(1,2)) nor to PRED (INT(3,4)) .

In terms of this set of predicates, Theorem 2.5 yields the following conclusion :

THEOREM 4.1 (1) PRED(MEET(A,B)) << SEC(PRED(A),PRED(B))
(2) PRED(JOIN(A,B)) >> UNI(PRED(A),PRED(B))

From a structural viewpoint , this theorem , with its "equality failure" , arises from the fact that , if we take

DEFINITION 4.1 PRED(I*(RE)) := SET(PRED(X) / X <* I*(RE))

the system STR(PRED(I*(RE)) , <<) is a sublattice of the larger system STR(PSET(PRED(RE)) , <<) , and the lattice operations MEET and JOIN correspond to the smaller system of the interval-sets of predicates STR(PRED(I*(RE)) , <<) (PSET is "powerset") .

Of course , the result of Theorem 4.1 , failing to provide an equality , stands across the straight on path from the semantics of modal intervals to the semantics of their inclusion-lattice . For a better interpretation of this difficulty , we shall consider , instead of the sets of predicates PRED(X) , some more restricted sets which will provide equality relations replacing the mere inclusions of Theorem 4.1 .

Let us define the sets of :

DEFINITION 4.2
Interval predicates as
PRED*(RE) := SET(x <* X' / X' <* I(RE))
Interval copredicates as
COPRED*(RE) := SET(x -<* X' / X' <* I(RE))
Interval predicates validated (or accepted) by A
PRED*(A) := SET(x <* X' / (x <* X') <* PRED(A))

Interval copredicates covalidated (or rejected) by A
COPRED*(A) := SET(x -<* X' / (x -<* X') <* COPRED(A))
where we say that P(.) is covalidated by A when P(.) <* COPRED(A) .

Now from Theorem 3.4 it follows :

THEOREM 4.2
(x -<* X') <* COPRED*(A) <==> (x <* X') <* PRED*(DUAL(A))

Moreover , the following theorem shows that the belonging relations of (x <* X') and of (x -<* X') , to the sets PRED*(A) and COPRED*(A) , are interval relations indeed :

THEOREM 4.3
(1) (x <* X') <* PRED*(A) <==> IMPR(X') << A
(2) (x -<* X') <* COPRED*(A) <==> PROP(X') >> A
where PROP(X') := (X' , E)
and IMPR(X') := (X' , U)

The statement (1) comes out from the left term being equivalent to SET(A) << X' when A is improper, and to SET(A) =* X' when A is proper . Statement (2) results from :
(x -<* X') <* COPRED(A) <==>
(x <* X') <* PRED(DUAL(A)) <==>
IMPR(X') << DUAL(A) <==>
PROP(X') >> A .

Theorem 4.3 suggests the identifications
(x <* X') <-----> IMPR(X')
(x -<* X') <-----> PROP(X')
PRED*(A) <-----> SET(IMPR(X') / IMPR(X') << A)
COPRED*(A) <-----> SET(PROP(X') / PROP(X') >> A) ;
and remark that "point-intervals" X' = INT(x1) can be identified to the predicates x = x1 or to the copredicates x -= x1 , according to their conventional membership to the proper or improper class of modal intervals .

Now , from these latter properties , the equalities missing in Theorem 4.1 for PRED(MEET(A,B)) and PRED(JOIN(A,B)) , which failed to establish a stronger association between the lattice of intervals and the lattice of interval-sets of predicates , are shown to hold in some cases , but not all , for interval predicates and copredicates :

THEOREM 4.4
(1) PRED*(MEET(A,B)) = SEC(PRED*(A) , PRED*(B))
(2) COPRED*(JOIN(A,B)) = SEC(COPRED*(A) , COPRED*(B))
(3) PRED*(JOIN(A,B)) >> UNI(PRED*(A) , PRED*(B))
(4) COPRED*(MEET(A,B)) >> UNI(COPRED*(A) , COPRED*(B))

About (1) , Theorem 4.3.(1) yields inmediatly that (x <* X') <* PRED(MEET(A,B)) ==> (x <* X') <* SEC(PRED*(A) , PRED*(B)) . The contrarywise inclusion comes from the lattice property (IMPR(X') << A , IMPR(X') << B) ==> IMPR(X') << MEET(A,B) . The assertion (2) is the dual statement of (1) and , moreover , results (3) and (4) are supported by obvious inclusion relations and by Theorems 2.5 and 3.5 . Moreover PRED*(JOIN(A,B)) can be larger than UNI(PRED*(A) , PRED*(B)) , as the example of (x=2.5) <* PRED*(JOIN(INT(1,2) , INT(2,4))) = PRED*(INT(1,4)) shows . All the same , COPRED*(MEET(A,B)) can be larger than UNI(COPRED*(A),COPRED*(B)) , as it comes out from the example (x-=2.5) <* COPRED*(MEET(INT(2,1),INT(4,3))) = COPRED*(INT(4,1)).

5.- CONCLUDING REMARKS.

Theorems 2.3, 2.5, 2.6, 3.1 and 3.5 , bring out the set-theoretical nature of the inclusion of modal intervals , since they tie modal intervals to the sets of predicates they accept (validate) or reject (covalidate) .

Theorems 4.1 and 4.4 , show that the intrinsic structure of the set of modal intervals , with their <<-meet and <<-join operations , does not allow a once for all association of modal intervals , neither with the whole set of the predicates they accept or reject , nor with the more specialized sets of interval-predicates or interval-copredicates .

Modal intervals are , indeed , intrinsically one-sided from the viewpoint of their association with sets of predicates upon the line of real numbers , as they can be identified with the set of interval-predicates they validate , A <----> PRED*(A) , only when interval predicates common to some family of modal intervals SET(A(i) / i<*I) are to be accounted for , in which case SEC(PRED*(A(i)) / i<*I) is equal to PRED*(MEET(A(i) / i<*I)) ; and they can be identified with the set of interval copredicates they reject , A <----> COPRED*(A) , only when interval copredicates common to some family of modal intervals do matter , in which case SEC(COPRED*(A(i)) / i<*I) is equal to COPRED*(JOIN(A(i) / i<*I)) .

Anyway , remark that all the inclusions of Theorems 4.1 and 4.4 become equalities when , between A and B , a relation A << B holds .

An application of the previous theory to the interpretation of interval-rounding results , from the viewpoint of the information they display , is the following theorem :

THEOREM 5.1

If DI << RE is a digital scale for the real numbers , and if outer and inner interval-rounding are defined by

OUT(INT(a,b)) :=
 ELEM(INT(a',b') / a'<*DI , b'<*DI , INT(a',b') >> INT(a,b))
INN(INT(a,b)) :=
 ELEM(INT(a',b') / a'<*DI , b'<*DI , INT(a',b') << INT(a,b))

then :

(1) PRED(INN(X)) << PRED(X)

(2) COPRED(OUT(X)) << COPRED(X)

(3) If the information supplied by some computing algorithm and/or some observation about a modal interval A is the pair of digital modal intervals A1 , A2 ,with A1 << A << A2 , then , the only predicates and copredicates that are A-decidable "a posteriori" are

the elements of PRED(A1) and of COPRED(A2) .

(4) With the same assumptions as in (3) , the "a priori" information induced by A onto A2 is PRED(A) , and , onto A1 , COPRED(A) .

As a particular application of this theorem to the case of ordinary intervals with the standard outwards rounding A2 >> A , only the "a priori" information PRED(A) (P(x) with E(x,A)P(x) ==> E(x,A2)P(x)) and the "a posteriori" information COPRED(A2) (P(x) with -E(x,A2)P(x) ==> -E(x,A) P(x) , or U(x,A2) -P(x) ==> U(x,A) -P(x)) are available .

The system of modal intervals can be used for actual computation by using the programming language SIGLA and the simulation language SIMSIGLA developed by the authors .

6.- BIBLIOGRAPHY.

Gardeñes E. , Trepat A. : "Fundamentals of SIGLA , an Interval Computing System on the Completed Set of Intervals" Computing 24 , Springer 1980 .

Gardeñes E. , Trepat A. , Janer J.M. : "SIGLA-PL/I , development and applications" Interval Mathematics 1980 , Ed. K. Nickel , Academic Press 1980 .

Gardeñes E. , Trepat A. , Janer J.M. : "Approaches to simulation and to the linear problem in the SIGLA System" Freiburger Intervall-Berichte 81/8 , Freiburg 1982 .

Gardeñes E. , Trepat A. , Mielgo H. : "Present perspective of the SIGLA Interval System" Freiburger Intervall-Berichte 82/9 , Freiburg 1982 .

Gardeñes E. : "Computing with the completed set of intervals : SIGLA-PL/I System" Proc. 22th Science Week , Damascus 1982 (to appear) .

Kaucher E. : "Interval analysis in the extended interval space IR" Computing Suppl. 2. ,Springer 1980 .

Nickel K. :"Verbandtheoretische Grundlagen der Intervall-Mathematik" Lecture Notes in Comp. Sc. 29 , Springer 1975 .